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Hypermultiplets and topological strings

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Abstract: The c-map relates classical hypermultiplet moduli spaces in compactifications of type II strings on a Calabi-Yau threefold to vector multiplet moduli spaces via a further compactification on a circle. We give an off-shell description of the c-map in $N = 2$ superspace. The superspace Lagrangian for the hypermultiplets is a single function directly related to the prepotential of special geometry, and can therefore be computed using topological string theory. Similarly, a class of higher derivative terms for hypermultiplets can be computed from the higher genus topological string amplitudes. Our results provide a framework for studying quantum corrections to the hypermultiplet moduli space, as well as for understanding the black hole wave-function as a function of the hypermultiplet moduli.

Keywords: Supergravity Models, Supersymmetric Effective Theories, Topological Strings.

Contents

1. Introduction

String theories with eight supercharges have been studied intensely over recent times, and still provide an excellent laboratory to study various phenomena that are important in any theory of quantum gravity. Perhaps the most studied examples are type II superstrings compactified on a Calabi-Yau threefold, and the connection to topological strings [1, 2]. In this context, it was recently conjectured that the partition functions of BPS black holes and topological strings are related in a simple way [3]. This conjecture was supported by the supergravity calculations done in [4], where subleading corrections to the BPS entropy coming from certain higher derivative terms in the supergravity effective action were determined.

While most aspects of black hole physics are related to the vector multiplet moduli space, there is also the hypermultiplet moduli space which is much less well understood. This is because the string coupling constant g_s sits in a hypermultiplet and therefore, this sector is subject to quantum corrections in g_s, both perturbatively and non-perturbatively through five-brane and membrane instantons [5]. Not much is known about these corrections, and this is mainly due to the complicated nature of the quaternion-Kähler (QK) geometry underlying the hypermultiplet moduli space. Some results can be obtained when restricting to the sector of the universal hypermultiplet only [6 − 9], or near a conifold singularity [10, 11].

At the classical level, however, the hypermultiplet moduli space is well known to be related to the special geometry of the vector multiplet sector through the c-map [12, 13]. At the supergravity level, the c-map arises upon compactifying the four-dimensional supergravity action on a circle, after which vector multiplets can be dualized into hypermultiplets. This means that the quaternionic geometry for the hypermultiplets is completely determined by a single function, namely the holomorphic prepotential of special geometry, or equivalently, the genus zero topological string amplitude. As will become clear in this paper, by using the conformal tensor calculus developed in [14, 15], the c-map has a natural and simple off-shell description in $N = 2$ projective superspace [16, 17]. We show that the superspace Lagrangian corresponding to the quaternionic geometry simply amounts to integrating the topological string amplitude over projective superspace. Moreover, we propose a relation between the higher genus topological string amplitudes and certain higher derivative terms in the effective action for the hypermultiplets. Similar terms were written down in components in [2]. Superspace effective actions, in relation to the c-map and topological strings, were also studied in [18, 19].

Initially, our motivation came from understanding stringy corrections to the hypermultiplet moduli space in terms of topological strings. Using superspace techniques, both perturbative and nonpertubative corrections can be encoded by a single function that determines the entire Lagrangian and quaternionic geometry [15]. This is illustrated in [7, 20] for the perturbative corrections. It remains to be understood if these corrections, and in particular the membrane instanton corrections, can also be calculated using topological strings. More recently, it was shown in [21] that there is also a connection to black hole physics. There, it was argued that the Hartle-Hawking wave function for BPS black holes should be understood as a function of all moduli that appear after a further compactification of the four-dimensional theory on a circle. These moduli naturally sit in hypermultiplets and therefore, the wave function is a function defined on the QK manifold, precisely as introduced by the c-map. See also [22]. We comment on these issues further in section 5.

This paper is organized as follows. In section 2, we review the c-map and describe the associated quaternionic geometries. In section 3, we state our main result, namely the superspace Lagrangian for the c-map, which we prove in section 4. We use various important geometrical features that are related to QK manifolds, namely their twistor spaces, and their hyperkähler cones. In section 5, we summarize our results and suggest possible further connections with black holes and topological strings. Finally, in section 6, we propose a class of higher derivative terms that should relate to higher genus topological string amplitudes.

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2. The c-map

In this section, we introduce our notation and review the c-map originally constructed in [12, 13].

Low-energy effective actions for type II strings on Calabi-Yau (CY) threefolds contain both vector multiplets and hypermultiplets. The $N = 2$ supergravity couplings require the scalars of the vector multiplets to parametrize a special Kähler manifold [23], whereas the hypermultiplet scalars parametrize a Quaternion-Kähler manifold [24].

The projective (or rigid) special Kähler manifold has real dimension $2(n+1)$ and is characterized by a holomorphic prepotential $F(X^I)$, which is homogeneous of degree two $(I = 1, \cdots, n+1)$. In type IIA (IIB) compactifications on a CY, we have $n = h_{1,1}\,(h_{1,2})$, respectively.

The Kähler potential and metric of the rigid special geometry are given by[1]

$$K = i(\bar{X}^I F_I - X^I \bar{F}_I) \ , \qquad N_{IJ} = i(F_{IJ} - \bar{F}_{IJ}) \ , \tag{2.1}$$

where F_I is the first derivative of F, etc. In terms of the periods of the Calabi-Yau manifold, for type IIB theory, we may identify

$$X^I = \int_{A_I} \Omega \ , \qquad F_I = \int_{B^I} \Omega \ , \tag{2.2}$$

where A_I and B^I are a real basis of three-cycles, $I = 0, ..., h_{1,2}$, and Ω is the holomorphic three-form. The Riemann bilinear identity implies that the Kähler potential (2.1) is

$$K = -i \int_{CY} \Omega \wedge \bar{\Omega} \ . \tag{2.3}$$

The (local) special Kähler geometry is then of real dimension $2n$, with complex inhomogeneous coordinates

$$Z^I = \frac{X^I}{X^1} = \{1, Z^A\} \ , \tag{2.4}$$

where A runs over n values. Its Kähler potential is given by

$$\mathcal{K} = \ln(Z^I N_{IJ} \bar{Z}^I) \ . \tag{2.5}$$

We further introduce the matrices [23]

$$\mathcal{N}_{IJ} = -i\bar{F}_{IJ} - \frac{(NX)_I (NX)_J}{(XNX)} \ , \tag{2.6}$$

where $(NX)_I \equiv N_{IJ} X^J$, etc. These matrices determine the gauge kinetic terms in the vector multiplet action and completely specify the couplings of vector multiplets to $N = 2$ supergravity in four spacetime dimensions.

[1] We use the modern conventions for the prepotential. In the original references [13, 25], different conventions were used: $K = \frac{1}{4}\left(X^I \bar{F}_I + \bar{X}^I F_I\right)$ and $N_{IJ} = \frac{1}{4}(F_{IJ} + \bar{F}_{IJ})$. It is straightforward to switch between these conventions.

The classical c-map is found by compactifying from four to three dimensions on a circle S^1. Each gauge field in four dimensions yields a pair of massless scalars in three dimensions: one comes from the component of the four dimensional gauge field along the circle, and the other from dualizing the remaining three-dimensional gauge field into a scalar.

Doing so, one maps a vector multiplet into a hypermultiplet, which we schematically denote by

$$\mathrm{c-map}: \quad (Z^A, A^I_\mu) \to (Z^A, A^I, A^I_{\hat{\mu}}) \to (Z^A, A^I, B_I) \ . \tag{2.7}$$

(Alternatively, we note that in three-dimensions, a vector multiplet is equivalent to a tensor multiplet. Then the c-map can be regarded as taking a vector multiplet from four to three dimensions and reinterpreting it as a tensor multiplet when returning to four dimensions. The tensor multiplet can then be dualized into a hypermultiplet in four dimensions.) In addition to the scalars arising from the gauge fields, two more scalars ϕ and σ come from the metric tensor, so we find a total of $4(n+1)$ scalars.

After the c-map, we obtain hypermultiplets whose scalars parametrize a target space that is a Quaternion-Kähler manifold of dimension $4(n+1)$. These spaces were described in [13], and further analyzed in [25]. We use the notation of the latter reference (replacing ϕ by e^ϕ). The QK metric can then be written as

$$\begin{aligned} \mathrm{d}s^2 =\ & \mathrm{d}\phi^2 - \mathrm{e}^{-\phi}(\mathcal{N}+\bar{\mathcal{N}})_{IJ} W^I \bar{W}^J + \mathrm{e}^{-2\phi}\Big(\mathrm{d}\sigma - \frac{1}{2}(A^I \mathrm{d}B_I - B_I \mathrm{d}A^I)\Big)^2 \\ & -4\mathcal{K}_{A\bar{B}}\, \mathrm{d}Z^A \mathrm{d}\bar{Z}^{\bar{B}} \ . \end{aligned} \tag{2.8}$$

The metric is only positive definite in the domain where $(ZN\bar{Z})$ is positive and hence $\mathcal{K}_{A\bar{B}}$ is negative definite. One can then show that $\mathcal{N}+\bar{\mathcal{N}}$ is negative definite [26]. The one-forms W^I are defined by

$$W^I = (\mathcal{N}+\bar{\mathcal{N}})^{-1\,IJ}\Big(2\bar{\mathcal{N}}_{JK}\mathrm{d}A^K - i\mathrm{d}B_J\Big) \ . \tag{2.9}$$

Although we have constructed the QK manifold from a supergravity action dimensionally reduced to three dimensions, one can write down four-dimensional supergravity lagrangians coupled to hypermultiplets that parametrize the same QK manifold as in (2.8). These are precisely the ones that appear in CY compactifications [12, 13]. The underlying mechanism is that T-duality

$$\mathrm{IIA}/\Big(\mathrm{CY}\times S^1_R\Big) \simeq \mathrm{IIB}/\Big(\mathrm{CY}\times S^1_{1/R}\Big) \tag{2.10}$$

relates type IIA and IIB string theory compactified on the *same* CY manifold.

Finally, the scalar field ϕ in (2.8) is identified with the dilaton and arises from the purely gravitational sector after the c-map. It belongs to the universal hypermultiplet. In our conventions, the relation with the string coupling constant is given by

$$g_s \equiv \mathrm{e}^{-\frac{1}{2}\phi_\infty} \ , \tag{2.11}$$

where ϕ_∞ is the value of the dilaton at infinity.

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3. Superspace description and Legendre transform

In this section we give the superspace description of the Lagrangian corresponding to the QK metric (2.8). Off-shell descriptions of matter couplings in $N = 2$ supergravity can be conveniently formulated using the superconformal tensor calculus. For hypermultiplets this tensor calculus was developed in [14, 15]. The geometry of the scalar manifolds is again projective, as for special Kähler manifolds. The compensators restoring the dilatations and $SU(2)_R$ inside the conformal group form an entire hypermultiplet. Adding the compensator to the original hypermultiplets that parametrize the $4(n+1)$-dimensional QK space, one obtains a parent space of dimension $4(n+2)$. This space is actually hyperkähler, admits a homothety and a $SU(2)$ isometry group that rotates the three complex structures. In the mathematics literature, this space is called the Swann space [27]. In the physics literature, we have used the name *hyperkähler cone* (HKC) [15].

3.1 Projective superspace

The lagrangian of an HKC corresponds to an $N = 2$ conformally invariant supersymmetric sigma model. Off shell, actions for such models can be conveniently written in terms of an integral in projective superspace [16, 17],

$$S = \mathrm{Im} \int \mathrm{d}^4x \, \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \, H(\eta, \zeta) \ , \tag{3.1}$$

where $\mathcal{C}$ is a contour in the complex ζ-plane that generically depends on the singularity structure of the superspace density H. We use the conventions for the $N = 2$ projective superfields η as in [15, appendix B].

The question is now which function H corresponds to the QK metric (2.8). Since for the tree-level c-map, there are (at least) $n+2$ commuting isometries generated by constant shifts of B_I and σ in (2.8), we can describe the action in terms of $N = 2$ tensor multiplets,

$$\eta^I = \frac{v^I}{\zeta} + G^I - \bar{v}^I \zeta \ , \tag{3.2}$$

where v^I project to chiral $N = 1$ superfields and G^I to linear superfields that describe $N = 1$ tensor multiplets; each of the latter contains a real scalar component field[2] as well as a component tensor. That tensor multiplets appear is no surprise–as noted above, the c-map maps vector multiplets directly into tensor multiplets, and the pure $N = 2$ supergravity multiplet is mapped into a double-tensor multiplet that is dual to the universal hypermultiplet. The component tensor multiplet Lagrangian that appears after the c-map was derived in [13]. More information on the double-tensor multiplet and the general $N = 2$ scalar tensor multiplet couplings can be found in [28]. Superspace effective actions in the context of type II string compactifications were also discussed in [18].

If we start with the IIB theory, the vector multiplet scalars are identified with the complex structure moduli, i.e., the periods of the holomorphic three-form (2.2). After the

[2] We will also denote the real scalar field of an $N = 1$ tensor multiplet G^I by G^I. It should be clear from the context what is meant.

c-map, on the type IIA side, the same periods now define scalars in the tensor multiplet sector. Moreover, the coordinates G^I are associated to the periods of the RR three-form C of type IIA. So we have

$$v^I = \int_{A_I} \Omega \ , \qquad G^I = \int_{A_I} C \ . \tag{3.3}$$

Similarly, there are the symplectically dual periods. Integrating C over the dual three-cycles B^I yields new scalars that are associated to the scalars dual to the tensor appearing in each tensor multiplet.

The constraints from superconformal invariance require scale and $SU(2)_R$ symmetry. This implies that H is a function homogeneous of first degree[3] (in η) and without explicit ζ dependence [15]. The scaling weights are such that η has weight two, and hence $H(\eta)$ has weight two as well. The $SU(2)_R$ transformations rotate the three scalars of each tensor multiplet:

$$\delta v^I = -i\varepsilon^3 v^I + \varepsilon^- G^I \ , \qquad \delta G^I = -2(\varepsilon^- \bar{v}^I + \varepsilon^+ v^I) \ , \tag{3.4}$$

and leave $G^I G^J + 2v^I \bar{v}^J + 2\bar{v}^I v^J$ invariant for all I and J.

As we will see later on, the c-map identifies the inhomogeneous vector multiplet scalars with the inhomogeneous coordinates $Z^I \equiv v^I/v^1$, and furthermore the coordinates A^I in (2.8) will be identified with the imhomogeneous coordinates associated to G^I. Notice the difference in scaling weights for the homogenous coordinates: the vector multiplet scalars X^I in (2.1) have conformal weight 1, whereas the v^I in (3.2) have weight two. This is the reason why we use a different symbol.

3.2 Legendre transform and hyperkähler potential

The projective superspace Lagrangian (3.1) defines the theory in terms of tensor multiplets. It is well known that in four spacetime dimensions, tensor multiplets can be dualized into hypermultiplets. In supersymmetric theories, such a duality can be performed by doing a Legendre transform on the $N = 1$ tensor multiplets [29, 30]. We first introduce the superspace Lagrangian density

$$\mathcal{L}(v, \bar{v}, G) \equiv \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta}\, H(\eta, \zeta) \ . \tag{3.5}$$

Integrating this over the $N = 2$ superspace measure, one gets the action (3.1).

The Legendre transform with respect to G^I is defined by

$$\chi(v, \bar{v}, w, \bar{w}) \equiv \mathcal{L}(v, \bar{v}, G) - (w + \bar{w})_I\, G^I \ , \qquad w_I + \bar{w}_I = \frac{\partial \mathcal{L}}{\partial G^I} \ . \tag{3.6}$$

Observe that the w_I have scaling weight zero, since both $\mathcal{L}$ and G have weight two. The object χ is called the hyperkähler potential, and serves as the Kähler potential for the HKC. It has scaling weight two and is a function of the complex coordinates v^I and w_I.

[3] Actually, quasihomogeneity up to terms of the form $\eta \ln(\eta)$ is sufficient[15], but such terms do not seem to arise in the c-map.

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The HKC metric only depends on w through the combination $w + \bar{w}$ and the absence of the imaginary parts of w reflects the commuting isometries. We can rewrite (3.6) as a Legendre transform on the function H,

$$\chi(v, \bar{v}, w, \bar{w}) = \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \Big[H\big(\frac{\bar{v}^I}{\zeta} + G^I - v^I \zeta\big) - G^I \frac{\partial H}{\partial G^I} \Big] \ , \tag{3.7}$$

with the defining relation for the coordinates $w + \bar{w}$,

$$(w + \bar{w})_I = \mathrm{Im} \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i \zeta} \frac{\partial H}{\partial \eta^I} \ . \tag{3.8}$$

Since H is homogeneous of first degree in η, it follows that the hyperkähler potential is also homogeneous of first degree in v and $\bar{v}$ in the sense of (we take λ real)

$$\chi(\lambda v, \lambda \bar{v}, w, \bar{w}) = \lambda \chi(v, \bar{v}, w, \bar{w}) \ . \tag{3.9}$$

The $SU(2)_R$ symmetry (3.4) after the Legendre transform acts on the coordinates v and w as [15]

$$\delta v^I = -i\varepsilon^3 v^I + \varepsilon^- G^I(v, \bar{v}, w, \bar{w}) \ , \qquad \delta w_I = \varepsilon^+ \frac{\partial \mathcal{L}}{\partial \bar{v}^I} \ , \tag{3.10}$$

where G^I has to be understood as the function of the coordinates $v, \bar{v}, w, \bar{w}$ obtained by the Legendre transform defined in (3.6). The coordinates w_I do not transform under ε^3. One can now explicitly check that the hyperkähler potential is $SU(2)_R$ invariant,

$$\delta\chi = \mathcal{L}_{v^I}\, \delta v^I + \mathcal{L}_{\bar{v}^I}\, \delta \bar{v}^I - \delta(w_I + \bar{w}_I)\, G^I = 0 \ . \tag{3.11}$$

(The δG terms cancel identically because χ is a Legendre transform). For the generators $\varepsilon^{\pm}$ this is immediately obvious; for variations proportional to ε^3 one needs to use the invariance of $\mathcal{L}$, i.e., $v^I \mathcal{L}_{v^I} = \bar{v}^I \mathcal{L}_{\bar{v}^I}$.

3.3 Hints from the rigid c-map

The Quaternion-Kähler space in the image of the c-map has dimension $4(n+1)$. The hyperkähler cone above it, which appears in the off-shell superspace formulation, has dimension $4(n+2)$. It therefore needs to be described by $n+2$ tensor multiplets, say η^I and η^0, where $I = 1, \cdots, n+1$. As we will show, the answer for the tree level c-map is given by

$$H(\eta^I, \eta^0) = \frac{F(\eta^I)}{\eta^0} \ , \tag{3.12}$$

where F is the prepotential of the special Kähler geometry, now evaluated on the tensor multiplet superfields η. This is our main result. Note that this H does not depend explicitly on ζ and is homogeneous of degree one, as required by superconformal invariance.

In the next section we give a detailed proof of (3.12) by explicit calculation; here we give the several intuitive arguments. In [12, 31, 15], the rigid c-map, a map from a $2(n+1)$-dimensional rigid special Kähler manifold with arbitrary (not necesserily homogeneous of second degree) holomorphic prepotential $F(X^I)$, to a $4(n+1)$-dimensional hyperkähler space is discussed. The c-map can again be formulated in terms of tensor multiplets, and the resulting projective superspace description is based on the function

$$\text{rigid c} - \text{map}: \qquad H(\eta^I, \zeta) = \frac{F(\zeta \eta^I)}{\zeta^2} \ . \tag{3.13}$$

The proof was given in [31, 15] by explicitly doing the contour integral with the contour around the origin. We give a more direct derivation in appendix A, where the rigid c-map is performed in $N = 1$ superspace.

When F is homogeneous of second degree, the ζ-dependence drops out of (3.13):

$$H = F(\eta^I) \ . \tag{3.14}$$

The ζ-independence is one of the two requirements of $N = 2$ superconformal symmetry. The other requirement is that H is homogenous of first degree (in η), but this is clearly not the case in (3.14). We can fix this problem by introducing a compensator η^0 that restores the correct homogeneity. The resulting hyperkähler space is then $4(n+2)$ dimensional, but this dimensionality is precisely what is needed for the *local* c-map ! So we are led to consider (3.12), which this defines an HKC above a QK manifold. All constraints from $N = 2$ superconformal symmetry are satisfied.

For the universal hypermultiplet, the corresponding holomorphic prepotential is quadratic, and there is only one (compensating) vector multiplet, hence $F(X^1) = i(X^1)^2$. After the c-map, we get a description in terms of two tensor multiplets:

$$H(\eta^0, \eta^1) = i \frac{(\eta^1)^2}{\eta^0} \ . \tag{3.15}$$

We know from explicit calculations in [15] that this is the right answer.

4. Hypermultiplet formulation

We now explicitly compute the QK geometry corresponding to the superspace density

$$H(\eta) = \frac{F(\eta^I)}{\eta^0} \ . \tag{4.1}$$

We show that the result matches exactly with the QK manifolds obtained by the c-map, i.e., we prove that (4.1) leads to (2.8).

4.1 Gauge fixing and contour integral

As explained in the previous section, any hyperkähler cone has a local $SU(2)_R$ and dilatation symmetry. In general, these symmetries are nonlinearly realized. On the tensor multiplet side, these symmetries act linearly: the $SU(2)_R$ rotates the three scalars inside each tensor multiplet, and the dilatations uniformly rescale them with weight two. The dilatations and $U(1) \subset SU(2)$ are usually combined together, as well as the remaining two generators (say $T_\pm$), into complex generators. To evaluate the contour integral, it is convenient to first impose some gauge choices. For the $T_\pm$ symmetries we choose the gauge

$$v^0 = 0 \ , \tag{4.2}$$

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where v^0 is the chiral $N=1$ superfield sitting in the compensator η^0. In this gauge, we have that $\eta^0 = G^0$ and this simplifies the pole structure in the complex ζ-plane. Note that this gauge choice does not impose restrictions on the periods (3.3), since in that formula the label I does not contain 0. The (complexified) dilatations can be fixed by choosing $v^1 = 1$, but we postpone implementing this gauge.

The contour integral that we now have to evaluate is given by

$$\mathcal{L}(v,\bar{v},G) = \frac{1}{G^0}\,\mathrm{Im}\oint \frac{\mathrm{d}\zeta}{2\pi i}\frac{F(\zeta\eta^I)}{\zeta^3}\ . \tag{4.3}$$

We have

$$\zeta\eta^I = v^I + \zeta G^I - \zeta^2\bar{v}^I\ , \qquad I = 1,\cdots,n+1\ , \tag{4.4}$$

which, for nonzero values of v, has no zeroes at $\zeta = 0$. Therefore, assuming F is regular at $\eta = 0$, $F(\zeta\eta)$ has no poles (in ζ) inside the contour around the origin (the same reasoning was used for the rigid c-map). It is now easy to evaluate the contour integral, because the residue at $\zeta = 0$ replaces all the $\zeta\eta^I$ by v^I. The result is

$$\mathcal{L}(v,\bar{v},G) = \frac{1}{4G^0}\Big(N_{IJ}G^IG^J - 2K\Big)\ , \tag{4.5}$$

where $K(v,\bar{v})$ is the Kähler potential of the rigid special geometry given in (2.1), with F_I now the derivative with respect to v^I . Because the v^I and G^I have conformal weight two, $F(v)$ and $K(v,\bar{v})$ have weight four, $\mathcal{L}$ has weight two and N_{IJ} has weight zero. The function $\mathcal{L}$ satisfies the Laplace-like equations [16, 17, 30]

$$\mathcal{L}_{G^IG^J} + \mathcal{L}_{v^I\bar{v}^J} = 0\ . \tag{4.6}$$

The equation is not satified for the components $\mathcal{L}_{G^0G^0}$ and $\mathcal{L}_{G^0G^I}$, because we have chosen a gauge, $v^0 = 0$. It would be interesting to compute $\mathcal{L}$ for arbitrary values of v^0. For the universal hypermultiplet, this was done in [7].

4.2 The hyperkähler potential

To compute the hyperkähler potential, we have to dualize the tensor multiplets into hypermultiplets. As described in the previous section, in supersymmetric theories, such a duality can be performed by doing a Legendre transform. Although this can be done in $N=2$ projective superspace, here we perform the duality in $N=1$ superspace by dualizing the $N=1$ tensor multiplets G^I into $N=1$ chiral superfields [29, 30]:

$$\chi(v,\bar{v},w,\bar{w}) = \mathcal{L}(v,\bar{v},G) + (w+\bar{w})_0\,G^0 - (w+\bar{w})_I\,G^I\ , \tag{4.7}$$

The hyperkähler potential χ, computed by extremizing[4] (4.7) with respect G^0, G^I completely determines the hypermultiplet theory and its associated hyperkähler geometry. In general, it is a function of the $2(n+2)$ complex coordinates v^0, v^I and w_0, w_I, but we have already gauge-fixed $v^0 = 0$. The geometry of the HKC only depends on w through the combination $w+\bar{w}$ which makes manifest the $n+2$ commuting isometries. The Legendre

[4] The relative minus signs between the last two terms in (4.7) is purely a matter of convention.

transform of (4.5) gives:

$$\frac{G^I}{G^0} = 2N^{IJ}(w+\bar{w})_J \ , \qquad (G^0)^2 = \frac{K}{2\Big((w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0\Big)} \ . \tag{4.8}$$

Up to an irrelevant overall sign, we find, using (4.5)

$$\chi\Big(v,\bar{v},G(v,\bar{v},w,\bar{w})\Big) = \frac{K(v,\bar{v})}{G^0} \ , \tag{4.9}$$

where G^0 is determined by (4.8). More explicitly, in terms of the HKC coordinates,

$$\chi(v,\bar{v},w,\bar{w}) = \sqrt{2}\ \sqrt{K(v,\bar{v})}\sqrt{(w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0} \ . \tag{4.10}$$

The reader might be surprised that in the HKC coordinates, the hyperkähler potential is proportional to the *square-root* of the special Kähler potential K. This is completely fixed by the scaling weights: The last factor on the right hand side has weight zero; since $K(v,\bar{v})$ has weight four (as opposed to $K(X,\bar{X})$ which has weight two), the square-root is needed to give χ scaling weight two. Similarly, for (4.9), the weights work out correctly because G^0 has weight two.

4.3 Twistor space

The twistor space above a $4(n+1)$ dimensional QK has dimension two higher, and is Kähler. It can be seen as a CP^1 bundle over the QK. It can also be obtained from the HKC by gauge fixing dilatations and $U(1) \subset SU(2)$ [15]. Instead of fixing a gauge, we use inhomogeneous coordinates Z^I as defined in (2.4). This allows us to choose different gauges. The Kähler potential of the rigid special Kähler manifold can be written as

$$K = |X^1|^2\, \mathrm{e}^{\mathcal{K}} \ , \tag{4.11}$$

with $\mathcal{K}$ given by (2.5). The same equation holds in the variables v^I, and we define inhomogeneous coordinates as

$$Z^I = \frac{v^I}{v^1} = \{1, Z^A\} \ , \tag{4.12}$$

where A runs over n values. As we show below, these inhomogeneous coordinates will be identified with (2.4).

The Kähler potential on the twistor space, denoted by K_T, is given by the logarithm of the hyperkähler potential [15]:

$$K_T(Z,\bar{Z},w,\bar{w}) = \frac{1}{2}\Big[\mathcal{K}(Z,\bar{Z}) + \ln\Big((w+\bar{w})_I N^{IJ}(w+\bar{w})_J - (w+\bar{w})_0\Big)\Big] + \ln(\sqrt{2}) \ . \tag{4.13}$$

On the twistor space, there always exists a holomorphic one-form $\mathcal{X}$ which can be constructed from the holomorphic two-form that any hyperkähler manifold admits. In our case this one-form is obtained from the holomorphic HKC two-form $\Omega = \mathrm{d}w_I \wedge \mathrm{d}v^I$. Without going into details, it is given by [15]

$$\mathcal{X} = 2Z^I \mathrm{d}w_I \ . \tag{4.14}$$

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The metric on the QK manifold can then be computed[5]:

$$G_{\alpha\bar{\beta}} = K_{T,\alpha\bar{\beta}} - \mathrm{e}^{-2K_T} \mathcal{X}_\alpha \bar{\mathcal{X}}_{\bar{\beta}} \ , \tag{4.15}$$

where the indices $\alpha, \beta = 1, \cdots, 2(n+1)$. The coordinates z^α on the QK consist of w_I, w_0 and the inhomogeneous coordinates Z^A of the special Kähler space. In total this gives $2(n+1) + 2 + 2n = 4(n+1)$–the (real) dimension of the QK.

4.4 The quaternionic metric

We now compute the QK metric that follows from the c-map using (4.15). To compare with (2.8) we only need to identify the coordinates w_I, w_0 with those of (2.8), since the Z^A coordinates of the special Kähler manifold can be identified with the ones above. We define

$$\begin{aligned} w_0 &= iA^I A^J F_{IJ} - i(\sigma + \frac{1}{2} A^I B_I) - \mathrm{e}^\phi \ , \\ w_I &= iF_{IJ} A^J - \frac{i}{2} B_I \ . \end{aligned} \tag{4.16}$$

The metric can be written in these coordinates, and after calculating we obtain the following result:

$$\begin{aligned} \mathrm{d}s^2 &= \mathrm{d}\phi^2 - \mathrm{e}^{-\phi} (\mathcal{N} + \bar{\mathcal{N}})^{-1\,IJ} \Big| 2\mathcal{N}_{IK} \mathrm{d}A^K + i \mathrm{d}B_I \Big|^2 \\ &\quad + \mathrm{e}^{-2\phi} \Big(\mathrm{d}\sigma - \frac{1}{2} (A^I \mathrm{d}B_I - B_I dA^I) \Big)^2 - 4\mathcal{K}_{A\bar{B}}\, \mathrm{d}Z^A \mathrm{d}\bar{Z}^{\bar{B}} \ . \end{aligned} \tag{4.17}$$

We have left out an overall normalization constant ($(-1/8)$ to be precise), and the matrix $\mathcal{N}_{IJ}$ is defined as in (2.6). How this matrix comes out of our calculation is somewhat nontrivial, and we have used the identity (see [25, appendix B])

$$(\mathcal{N} + \bar{\mathcal{N}})^{-1\,IJ} = N^{IJ} - \frac{X^I \bar{X}^J + \bar{X}^I X^J}{(XN\bar{X})} \ . \tag{4.18}$$

The two terms on the right hand side follow from the two terms on the right hand side of (4.15). It is then easy to see that this metric coincides with (2.8). This concludes the proof of (4.1).

Finally, from (4.8) we can easily derive the following relations between the QK coordinates and the tensor multiplet coordinates:

$$2A^I = \frac{G^I}{G^0} \ , \qquad 4\mathrm{e}^\phi = \frac{K}{(G^0)^2} \ . \tag{4.19}$$

This associates directly the coordinates A^I with the periods as defined in (3.3). This is consistent with the supergravity analysis in [25] where it was shown that the scalars (A^I, B_I) form a symplectic pair, whereas the dilaton is an invariant.

[5] Note that the constant term in K_T (4.13) enters in (4.15).

5. Relation to black holes and topological strings

5.1 Recap

What we have established in the previous sections is that the tensor multiplet Lagrangian at tree-level (in the string coupling constant g_s) is determined by the prepotential $F(X^I)$ of special geometry via the c-map. The action can be written in projective superspace as

$$S_{\text{tensor}} = \text{Im}\int \mathrm{d}^4x\ \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\ \oint_{\mathcal{C}} \frac{\mathrm{d}\zeta}{2\pi i\zeta}\ \frac{F(\eta^I)}{\eta^0}\ , \tag{5.1}$$

where $\eta(\zeta)$ are $N=2$ tensor multiplets defined in (3.2), consisting of an $N=1$ chiral multiplet v and an $N=1$ tensor multiplet G. The action (5.1) has an $SU(2)_R$ symmetry, and in the gauge $v^0=0$ the contour integral can be done most easily:

$$S_{\text{tensor}} = \int \mathrm{d}^4x\ \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\ \frac{1}{4G^0}\Big(N_{IJ}G^IG^J - 2K(v,\bar{v})\Big)\ . \tag{5.2}$$

To get the tree-level hypermultiplet action, we need to Legendre transform (5.2) with respect to the $N=1$ tensor multiplets G^I and G^0. The resulting function was given in (4.10) and is called the hyperkähler potential χ. It can be written in a more compact form as

$$\chi\Big(v,\bar{v},G(v,\bar{v},w,\bar{w})\Big) = \frac{K(v,\bar{v})}{G^0}\ , \tag{5.3}$$

where $G^0(v,\bar{v},w,\bar{w})$ is determined by (4.8). One can now make use of the homogeneity properties of the hyperkähler potential (3.9). Introducing the *weight one* coordinates

$$X^I(v,\bar{v},w,\bar{w}) \equiv \frac{v^I}{\sqrt{G^0(v,\bar{v},w,\bar{w})}}\ , \tag{5.4}$$

we can conveniently rewrite the hyperkähler potential as

$$\boxed{\chi(v,\bar{v},w,\bar{w}) = K\Big(X^I(v,\bar{v},w,\bar{w}),\bar{X}^I(v,\bar{v},w,\bar{w})\Big)}\ . \tag{5.5}$$

Here K is the Kähler potential of the rigid special geometry, $K(X,\bar{X}) = i(\bar{X}^IF_I - X^I\bar{F}_I)$ and the X^I scale with weight one. There is thus a very simple rule to obtain the HKC hyperkähler potential from the special Kähler potential: just replace the holomorphic coordinates X^I by the functions $X^I(v,\bar{v},w,\bar{w})$ as defined by (5.4)!

The relations (5.4) and (5.5) are written in an $N=1$ or component language. The corresponding relations in $N=2$ language can best be formulated in terms of tensor multiplets, and are simply the statement that the function H is related to the prepotential F as in (4.1). Indeed, as in (5.4), if we define *weight one* $N=2$ multiplets

$$X^I(\eta) \equiv \frac{\eta^I}{\sqrt{\eta^0}}\ , \tag{5.6}$$

we can write the tensor multiplet superspace density H as

$$\boxed{H(\eta^I,\eta^0) = F[X^I(\eta^I,\eta^0)]}\ . \tag{5.7}$$

Notice that both formulas (5.5) and (5.7) are consistent with the scaling weights and homogeneity properties. There is still a (complexified) dilatation gauge that we could choose, for instance $G^0 = 1$, or $v^1 = 1$, but we prefer not do so to keep our formulas are valid in any gauge. Furthermore, note that the hyperkähler potential χ is a scalar function under symplectic transformations induced by the vector multiplet theory. Under symplectic transformations, (X^I, F_I) transforms linearly under the symplectic group, such that K is a scalar. On the other hand, H is not a scalar, but transforms like the prepotential F.

5.2 Legendre transform and black holes

We now repeat some aspects of the Legendre transform as done in [3], see also [32, 21]. Consider the identity

$$-\frac{1}{2}|\lambda|^2 K(X, \bar{X}) = \mathrm{Im}\Big(\lambda^2 X^I F_I\Big) - 2\,\mathrm{Im}(\lambda X^I)\,\mathrm{Re}(\lambda F_I)\ , \tag{5.8}$$

for any complex quantity λ. Using the homogeneity property of F, we can rewrite this identity as

$$-\frac{1}{4}|\lambda|^2 K(X, \bar{X}) = \mathrm{Im}[\lambda^2 F(X^I)] - \,\mathrm{Im}(\lambda X^I)\,\mathrm{Re}(\lambda F_I(X^I))\ . \tag{5.9}$$

We now define real quantities p^I, ϕ^I, q^I and $\mathcal{F}$ by

$$p^I + \frac{i}{\pi}\phi^I \equiv \lambda X^I\ , \quad q_I \equiv \mathrm{Re}(\lambda F_I(X^I))\ , \quad \mathcal{F} \equiv -\pi \mathrm{Im}[\lambda^2 F(X^I)] = -\pi \mathrm{Im}[F(\lambda X^I)]\ . \tag{5.10}$$

They satisfy the relation

$$q_I = -\frac{\partial \mathcal{F}}{\partial \phi^I}\ , \tag{5.11}$$

which can be used to write the ϕ^I as a function of q_I and p^I. Now it follows that the Kähler potential is the Legendre transform of $\mathcal{F}$:

$$\frac{\pi}{4} K(p, q) = \mathcal{F}[p, \phi(p, q)] + \phi^I(p, q)\, q_I\ . \tag{5.12}$$

Notice that we have not used BPS-like equations whatsoever to derive this relation. In the context of black holes, the q_I and p^I are of course related to the electric and magnetic charges of the black hole via the attractor equations.

Using (5.5), we can now write the hyperkähler potential as a Legendre transform,

$$\frac{\pi}{4}\,\chi(p, q) = \mathcal{F}[p, \phi(p, q)] + \phi^I(p, q) q_I\ . \tag{5.13}$$

The only thing we have to do in this equation is to interpret the "charges" p^I and q_I in terms of the hypermultiplet variables, i.e.,

$$p^I + \frac{i}{\pi}\phi^I = \frac{\lambda v^I}{\sqrt{G^0(v, \bar{v}, w, \bar{w})}}\ , \tag{5.14}$$

and similarly for q_I.

Could this have any relation with the Legendre transform as defined in (3.6)? As they stand, the two Legendre transforms seem totally different, since in (3.6) one Legendre transforms $\mathcal{L}$ with respect to the tensor multiplet variables G^I, whereas in (5.13) one Legendre transforms $\mathcal{F}$ with respect to the $\mathrm{Im}(X^I)$. On the other hand, the relations (5.5) and (5.7) do connect them. Furthermore, the $SU(2)_R$ transformations rotate G^I into $\mathrm{Im}(v^I)$; since we work in the gauge $v^0 = 0$, this symmetry is not manifest, but it should allow us to rotate the two Legendre transforms into each other.

We now give an additional argument. As discussed in [21] (which involves an interpretation of the OSV wave function in the black hole context [3]) the Hartle-Hawking wave function for black holes is a function, not only of the moduli of the Calabi-Yau, but also of the Ramond-Ramond gauge potentials, which in this paper we have denoted by G^I. However, the notion of the mini-superspace used in [21] amounts to choosing a reduction to the BPS sector of the theory. In such a case G^I (to be more precise, $G^I/\sqrt{G^0}$) gets identified with $\mathrm{Im}X^I$. So in principle we can view the topological string wave function as a function of $\mathrm{Re}X^I + iG^I$. Using the relations between the topological string amplitude and the prepotential $\lambda X^1 = \frac{4\pi i}{g_{\rm top}}$ and $\mathcal{F} = F_{\rm top} + \bar{F}_{\rm top}$, we can write

$$\psi_{\text{black hole}} = \mathrm{e}^{F_{\rm top}\left[\mathrm{Re}X^I + iG^I\right]} \ . \tag{5.15}$$

In this case in order to obtain the black hole entropy we have to consider the Fourier transform (which to leading order is the Legendre transform) of $|\psi|^2$ with respect to G^I. This then is exactly of the same structure as in (3.6) or (4.7). The question is then why should a formula resembling the formula corresponding to entropy of black holes be related to our discussion here.

Here we offer a possible explanation, which may be the basis of this connection. As discussed in [21], the relation between the Hartle-Hawking wave function and topological strings goes via compactification of the four dimensional theory on a circle and writing the reduced wave function on possible degrees of freedom, subject to preserving half the supersymmetry. However, once we compactify on a circle the black hole states, running in the extra circle, viewed as Euclidean time, play the role of instantons of the three dimensional theory, which by T-duality on the radius of the circle gets related to hypermultiplets in the 4-dimensional theory, as discussed in the context of c-map. Thus the exchange of the role between black hole states and instantons may be a partial explanation of this fact.

6. Higher derivative terms

A natural question is to ask what the higher genus partition function of the topological string computes for the tensor or hypermultiplet. This question was addressed in [2], where higher derivative corrections on the universal hypermulitplet were found that multiply the genus g partition function. Here we will write down such terms in superspace by using a similar procedure as for the vector multiplet action.

The topological A-model computes F-terms in the four-dimensional supergravity effective action, proportional to higher powers of the Riemann curvature and graviphoton field strength [2, 1]. They can be nicely encoded using superspace techniques, by putting the vector multiplets in a chiral background [33]. In $N = 2$ chiral superspace, vector multiplet actions can be written as

$$S = \mathrm{Im}\Big(\int \mathrm{d}^4x\, \mathrm{d}^4\theta F(X) \Big) \ , \tag{6.1}$$

where F is a holomorphic function, homogeneous of degree two in $N = 2$ restricted chiral superfields X^I (i.e., the $N = 2$ vector multiplet superfields). To generate the higher curvature terms, one considers generalized actions of the type (6.1) by including a background $N = 2$ (unrestricted) chiral superfield Φ, which is associated with the square of the Weyl multiplet $\mathcal{W}^2$ (having scaling weight 2). The action is based on a new prepotential $F(X, \mathcal{W}^2)$ and we expand it in a power series

$$F(X^I, \mathcal{W}^2) = \sum_{g=0}^{\infty} F_g(X^I)(\mathcal{W}^2)^g \ . \tag{6.2}$$

It then turns out that the coefficient functions $F_g(X)$ are related to the genus g topological partition function.

We can now set up a similar construction on the tensor multiplet side, after having done the c-map. The Weyl multiplet becomes the universal hypermultiplet after the c-map, so one indeed expects the higher genus terms to correspond to higher derivative terms in the universal hypermultiplet only [2]. We can immitate the same trick as for the vector multiplets, by putting the Lagrangian corresponding to (4.1) in an (unrestricted) projective superfield background Υ. Based on the c-map, and arguments given above, we put the prepotential in this background and expand

$$F(\eta^I, \Upsilon) = \sum_{g=0}^{\infty} F_g(\eta^I)(\Upsilon)^g \ . \tag{6.3}$$

A similar expansion also appears in [18]. The background Υ should be identified with the c-map of the (square of the) Weyl multiplet $\mathcal{W}^2$, and is describing the universal hypermultiplet. The coefficient functions F_g are again be related to the genus g partition function of the topological string. We take

$$\Upsilon = \nabla^2 \bar{\nabla}^2 \, (L^2)^{1/2} \ , \tag{6.4}$$

where L^2 is an appropriate function of the tensor multiplet describing the universal hypermultiplet. The ∇ and $\bar{\nabla}$ operators are there to generate the higher derivatives, in such a way that (powers of) Υ can be integrated over superspace, and such that the the scaling and $SU(2)_R$ symmetries are preserved. A candidate would be

$$L^2 = L^I_{ij} N_{IJ} L^{J\, ij} \ , \tag{6.5}$$

where the L_{ij} describe the components of a tensor multiplet η by means of $L_{+-} \propto G, L_{++} \propto v$. That this multiplet contains the dilaton can be argued from (4.19). It remains to be shown that this leads to the same answer as in [2], where higher derivative terms for the hypermultiplets were written down in components. A more systematic treatment of higher derivative terms for tensor multiplets in components will be given in [34].

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A. The rigid c-map

The classical rigid c-map can be easily understood by reducing $N = 2$ $D = 4$ superspace to $N = 1$ superspace and then going down to $D = 3$; the relation is also direct and transparent in $N = 2$ superspace, but the projective formalism is less familiar [31, 15].

Consider an $N = 2$ vector-multiplet Lagrange density:

$$\mathcal{L}_{vect} = -\mathrm{Im}\left[\int \mathrm{d}^4\theta F(X^I)\right] , \tag{A.1}$$

where the measure $\mathrm{d}^4\theta$ is the $N = 2$ chiral measure. In $N = 1$ superspace, this becomes

$$\mathcal{L}_{vect} = -\mathrm{Im}\left[\int \mathrm{d}^2\theta \left(\frac{\partial F}{\partial X^I}\bar{D}^2\bar{X}^I + \frac{\partial^2 F}{\partial X^I \partial X^J} W^{I\alpha} W^J_\alpha\right)\right] , \tag{A.2}$$

where X^I are now $N = 1$ chiral superfields and $W^{I\alpha}$ is the $N = 1$ vector multiplet field strength. Descending to to $D = 3$ does not change (A.2) *except* that $W^{I\alpha}$ can now be written in terms of a real lower dimension field strength G^I:

$$W^I_\alpha = \frac{i}{\sqrt{2}}\bar{D}_\alpha G^I \quad , \quad D^2 G^I = \bar{D}^2 G^I = 0 ; \tag{A.3}$$

this is not possible in $D = 4$ because it violates Lorentz invariance–W_α and $\bar{D}_{\dot\alpha}$ transform in conjugate representations of $Sl(2, \mathbb{C})$ that reduce to the same representation of $Sl(2, \mathbb{R})$. Thus we find the $D = 3$ Lagrange density:

$$\mathcal{L}_{vect} = \mathrm{Im}\left[\int \mathrm{d}^2\theta \left(-\frac{\partial F}{\partial X^I}\bar{D}^2\bar{X}^I + \frac{1}{2}\frac{\partial^2 F}{\partial X^I \partial X^J}\bar{D}^\alpha G^I \bar{D}_\alpha G^J_\alpha\right)\right] . \tag{A.4}$$

Using the chirality constraints on X^i and the linear constraints (A.3) on G, we can rewrite this as a full superspace integral:

$$\mathcal{L}_{vect} = \int \mathrm{d}^2\theta\, \mathrm{d}^2\bar{\theta}\ \mathrm{Im}\left(-\frac{\partial F}{\partial X^I}\bar{X}^I + \frac{1}{2}\frac{\partial^2 F}{\partial X^I \partial X^J} G^I G^J\right) . \tag{A.5}$$

We now compare this to the hypermultiplet action. We use the tensor multiplet projective superspace description of the $D = 4$ hypermultiplet; this involves superfields $\eta^I = \frac{X^I}{\zeta} + G^I - \zeta\bar{X}^I$ which are real under the composite operation of complex conjugation and the antipodal map $\bar{\zeta} \to \frac{-1}{\zeta}$ on $\mathbb{CP}^1$. The general tensor multiplet $D = 4$ Lagrange density is

$$\mathcal{L}_{hyper} = \int d^2\theta\, d^2\bar{\theta} \oint \frac{d\zeta}{2\pi i \zeta} G(\eta^I, \zeta) \quad , \quad \mathcal{R}(G) = G \ . \tag{A.6}$$

Now consider the special case when

$$G = \mathrm{Im}_{\mathcal{R}} \left(\frac{F(\zeta\eta^I)}{\zeta^2} \right) \equiv -i \left[\frac{F(\zeta\eta^I)}{\zeta^2} - \zeta^2 \bar{F}(\frac{-\eta^I}{\zeta}) \right] \ , \tag{A.7}$$

where $\mathrm{Im}_{\mathcal{R}}$ means the imaginary part with respect to the composite conjugation $\mathcal{R}$. Consider the first term

$$\frac{F(\zeta\eta^I)}{\zeta^2} \equiv \frac{F(X^I + \zeta G^I - \zeta^2 \bar{X}^I)}{\zeta^2} \ ; \tag{A.8}$$

for $F(X^I)$ regular at $X^I = 0$, the contour integral gets just two contributions:

$$-\frac{\partial F}{\partial X^I} \bar{X}^I + \frac{1}{2} \frac{\partial^2 F}{\partial X^I \partial X^J} G^I G^J \ . \tag{A.9}$$

Plugging this into (A.6) gives precisely $\mathcal{L}_{vect}$ (A.5).

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